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# Characterizations of Local Best Chebyshev Approximations

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Local best Chebyshev approximations are characterized by a condition which can be considered as a generalization of the general Kolmogoroff criterion. In the discrete case, this condition is shown to be equivalent to the general Kolmogoroff criterion itself. Finally, local characterizations of the general Kolmogoroff criterion are given.

## 1. FORMULATION OF THE PROBLEM

Let $B \subset \mathbb{R}^n$ be a compact set and $C(B)$ be the vector space of all continuous real-valued functions on $B$ equipped with the maximum norm

$$\| g \| = \max_{x \in B} | g(x)|, \qquad g \in C(B).$$

Let further $A$ be a nonempty subset of $\mathbb{R}^p$ and $F: A \to C(B)$ be a given continuous mapping. If $f \in C(B)$ is given, we say $F(\hat{a})$, $\hat{a} \in A$, is a "(strict) best approximation" to $f$ on $B$ in the Chebyshev sense if

$$\| f - F(\hat{a})\| \underset{(<)}{\leqslant} \| f - F(a)\| \qquad \forall\, a \in A \setminus \{\hat{a}\}. \tag{1.1}$$

Similarly, $F(\hat{a})$ is a "(strict) local best approximation" to $f$ on $B$ if there is an $\varepsilon > 0$ such that

$$\| f - F(\hat{a})\| \underset{(<)}{\leqslant} \| f - F(a)\| \qquad \forall\, a \in A \cap U_{\hat{a}}^{\varepsilon}$$

for

$$U_{\hat{a}}^{\varepsilon} = \{a \in \mathbb{R}^p \mid \| \hat{a} - a \| < \varepsilon\} \setminus \{\hat{a}\}.$$



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We require now that $f$ does not lie in $F(A)$, the image of $A$ under $F$, so that $\|f-F(a)\|>0$ for all $a \in A$.

We will use the set of extremal points at $a \in A$,

$$I(a)=\{x \in B \mid |f(x)-F(a,x)|=\|f-F(a)\|\}$$

and the following sign function:

$$\sigma(a,x)=\begin{cases}\operatorname{sgn}(f(x)-F(a,x)) & \text{if } f(x)-F(a,x) \neq 0\\ 0 & \text{if } f(x)-F(a,x)=0\end{cases},$$

where $(a,x) \in A \times B$. Obviously, $\sigma(a,x)$ is different from zero for all $x \in I(a)$ with arbitrary $a \in A$.

Finally, we define for each compact subset $D$ of $B$

$$\|g\|_D=\max_{x \in D}|g(x)|, \qquad g \in C(B).$$

## 2. Introduction

It has been known for some time that for $F$ being linear, $F(\hat{a})$ is a best approximation to $f$ if and only if the general Kolmogoroff criterion

$$\min_{x \in I(\hat{a})}(f(x)-F(\hat{a},x))(F(a,x)-F(\hat{a},x)) \leqslant 0 \qquad \forall\, a \in A$$

or, equivalently,

$$\min_{x \in I(\hat{a})} \sigma(\hat{a},x)(F(a,x)-F(\hat{a},x)) \leqslant 0 \qquad \forall\, a \in A \tag{2.1}$$

is satisfied, and further, that in the general case (2.1) represents usually only a sufficient condition for a best approximation [6]. A great deal of effort has been made in characterizing those sets of functions in which each best approximation is characterized by (2.1). We refer, for instance, to [1] and [5]; for further references, see also [7].

That (2.1) is a sufficient and, furthermore, very restrictive condition for $F(\hat{a})$ to be a best approximation becomes obvious from the following two lemmas.

Lemma 2.1. *For each $a \in A$, let*

$$K(a)=\{x \in I(\hat{a}) \mid \|f-F(\hat{a})\| \leqslant |f(x)-F(a,x)|\}.$$

*Then* (2.1) *holds true if and only if for each $a \in A$ the following two conditions are satisfied:*

(i) $\|f-F(\hat{a})\| \leqslant \max_{x \in I(\hat{a})} |f(x)-F(a,x)|$,

(ii) $\sigma(a,x)=\sigma(\hat{a},x)$ *for at least one* $x \in K(a)$.

*Proof.* Equation (2.1) is interchangeable with

$$\max_{x \in I(\hat{a})} \sigma(\hat{a},x)\left[(f(x)-F(a,x))-(f(x)-F(\hat{a},x))\right] \geqslant 0 \qquad \forall\, a \in A$$

which again is equivalent to

$$\|f-F(\hat{a})\| \leqslant \max_{x \in I(\hat{a})} \sigma(\hat{a},x)(f(x)-F(a,x)) \qquad \forall\, a \in A.$$

The last inequality is true if and only if conditions (i) and (ii) hold at the same time.

Hence (2.1) implies

$$\|f-F(\hat{a})\|_{I(\hat{a})} \leqslant \|f-F(a)\|_{I(\hat{a})} \qquad \forall\, a \in A. \tag{2.2}$$

(2.2) is equivalent to the following generalization of the general Kolmogoroff criterion [4]:

$$\min_{x \in I(\hat{a})} \{(f(x)-F(\hat{a},x))(F(a,x)-F(\hat{a},x))-\tfrac{1}{2}(F(a,x)-F(\hat{a},x))^2\} \leqslant 0 \qquad \forall\, a \in A. \tag{2.3}$$

(Another proof of this equivalence is included in the proof of Theorem 3.2 below.) That (2.2) or (2.3), respectively, is sufficient for (1.1) to hold, follows from the next lemma.

LEMMA 2.2. *If* $F(\hat{a})$ *is a best approximation to* $f$ *on the set of extremal points at* $\hat{a} \in A$, *then* $F(\hat{a})$ *is also a best approximation to* $f$ *on* $B$.

*Proof.* $\|f-F(\hat{a})\| = \|f-F(\hat{a})\|_{I(\hat{a})} \leqslant \|f-F(a)\|_{I(\hat{a})} \leqslant \|f-F(a)\| \;\forall\, a \in A$.

It has further been known that likewise the general Kolmogoroff criterion confined to a neighborhood of $\hat{a}$ is usually only a sufficient condition for a local best approximation [3, 6]. In [6] it was proved that it becomes also a necessary condition for a local best approximation $F(\hat{a})$ if $F(\hat{a})$ is a best approximation to $f$ not only on $B$ but also on $I(\hat{a})$, i.e., shortly if (2.2) is satisfied. (This fact will be proved later in another manner.)

From our discussion then follows already that (2.1), (2.2) and (2.3) are locally equivalent, and that, therefore, (2.2) or (2.3), respectively, considered in an appropriate neighborhood of $\hat{a}$ is likewise, in general only a sufficient condition for $F(\hat{a})$ to be a local best approximation. (See also Theorem 3.2 below.)

Since (2.1) and (2.2) are locally equivalent, we can conclude from Lemma

2.1 that a certain sign condition is locally always fulfilled. Using this fact to advantage, we can characterize local best approximations in a nontrivial way under very general circumstances.

Finally, we want to mention that in [7] a survey of characterizations of local best Chebyshev approximations under conditions of first and second order was given.

## 3. Characterizations of Local Best Approximations

THEOREM 3.1. *Let $D$ be an arbitrary compact subset of $B$ which contains at least one element of $I(\hat{a})$ and let $\rho = \|f - F(\hat{a})\|$. Let further*

(i) $\|f - F(\hat{a})\|_D \underset{(<)}{\leqslant} \|f - F(a)\|_D \qquad \forall\, a \in A \cap U_{\hat{a}}^{\varepsilon}$.

(ii) $\inf_{x \in D} \{\rho - \sigma(\hat{a}, x)(f(x) - F(\hat{a}, x)) + \sigma(\hat{a}, x)(F(a, x) - F(\hat{a}, x))\} \underset{(<)}{\leqslant} 0$ *for all $a \in A \cap U_{\hat{a}}^{\delta}$, where the infimum is achieved for an $x \in D$.*

(iii) *With $f_\lambda = F(\hat{a}) + \lambda(f - F(\hat{a}))$, for all $\lambda > 0$*

$$\inf_{x \in D} \{\rho - \sigma(\hat{a}, x)(f(x) - F(\hat{a}, x)) + |f_\lambda(x) - F(\hat{a}, x)| - |f_\lambda(x) - F(a, x)|\} \underset{(<)}{\leqslant} 0 \qquad \forall\, a \in A \cap U_{\hat{a}}^{\delta},$$

*where for $a \in A \cap U_{\hat{a}}^{\delta}$ fixed, the infimum is achieved for all $\lambda > 0$ at an $x_a \in D$ with $\sigma(\hat{a}, x_a) \neq 0$; i.e., $x_a$ is independent of $\lambda$.*

*Then the following relations hold between* (i) *and* (iii):

(i) $\Rightarrow$ (ii) *with a certain $\delta$, $0 < \delta \leqslant \varepsilon$,* (ii) $\Rightarrow$ (i) *with $\varepsilon = \delta$, and* (ii) $\Leftrightarrow$ (iii), *i.e., a neighborhood of $\hat{a}$ exists in which* (i), (ii) *and* (iii) *are equivalent.*

*Proof.* (i) $\Rightarrow$ (ii): Part (i) implies that for each $a \in A \cap U_{\hat{a}}^{\varepsilon}$ there is an $x_a \in D$ with $|f(x_a) - F(a, x_a)| = \|f - F(a)\|_D$ such that

$$\rho \underset{(<)}{\leqslant} |f(x_a) - F(\hat{a}, x_a) - (F(a, x_a) - F(\hat{a}, x_a))|. \tag{3.1}$$

Because of the continuity of the mapping $F: A \to C(B)$ there is now a $\delta_\rho > 0$ for $\rho$, such that

$$\|F(\hat{a}) - F(a)\| < \rho \qquad \forall\, a \in A \cap U_{\hat{a}}^{\delta_\rho}. \tag{3.2}$$

Therefore, (3.1) yields for $a \in A \cap U_{\hat{a}}^{\delta}$, $\delta = \min(\varepsilon, \delta_\rho)$, that $\sigma(\hat{a}, x_a) \neq 0$ and

$$\sigma(\hat{a}, x_a)(F(a, x_a) - F(\hat{a}, x_a)) \leqslant 0. \tag{3.3}$$

For if we assume that both terms $(f(x)-F(\hat{a},x))$ and $(F(a,x_a)-F(\hat{a},x_a))$ are positive or negative at the same time, we get employing (3.2) the following contradiction:

$$\rho \leqslant |f(x_a)-F(\hat{a},x_a)-(F(a,x_a)-F(\hat{a},x_a))|$$

$$= \left\{ \begin{array}{l} \text{either} \\ |f(x_a)-F(\hat{a},x_a)| - |F(a,x_a)-F(\hat{a},x_a)| \\ \text{or} \\ |F(a,x_a)-F(\hat{a},x_a)| - |f(x_a)-F(\hat{a},x_a)|. \end{array} \right\} < \rho$$

By virtue of (3.3), we are now able to write (3.1) as

$$\rho \underset{(<)}{\leqslant} |f(x_a)-F(a,x_a)| = |f(x_a)-F(\hat{a},x_a)-(F(a,x_a)-F(\hat{a},x_a))|$$

$$= \sigma(\hat{a},x_a)(f(x_a)-F(\hat{a},x_a)) - \sigma(\hat{a},x_a)(F(a,x_a)-F(\hat{a},x_a)). \tag{3.4}$$

Since further for arbitrary $x \in D$

$$\sigma(\hat{a},x)(f(x)-F(\hat{a},x)) - \sigma(\hat{a},x)(F(a,x)-F(\hat{a},x)) \leqslant |f(x)-F(a,x)|$$
$$\leqslant |f(x_a)-F(a,x_a)| \tag{3.5}$$

is true, we can conclude condition (ii) from (3.4) and (3.5)

(ii) ⇒ (i): If, conversely, (ii) is given then for each $a \in A \cap U_{\hat{a}}^{\delta}$ there is an $x_a \in D$ such that

$$\rho - \sigma(\hat{a},x_a)(f(x)-F(\hat{a},x_a)) + \sigma(\hat{a},x_a)|(f(x)-F(\hat{a},x_a))$$
$$-(f(x)-F(a,x_a))| \underset{(<)}{\leqslant} 0$$

or, equivalently,

$$\rho \underset{(<)}{\leqslant} \sigma(\hat{a},x_a)(f(x)-F(a,x_a)).$$

The last inequality implies (i) with $\varepsilon = \delta$.

(ii) ⇒ (iii): Let the infimum in (ii) be achieved for $a \in A \cap U_{\hat{a}}^{\delta}$ at $x_a \in D$. Then we have for all $\lambda > 0$

$$\rho - \sigma(\hat{a},x_a)(f(x_a)-F(\hat{a},x_a)) + \sigma(\hat{a},x_a)|f_\lambda(x_a)-F(\hat{a},x_a)$$
$$-(f_\lambda(x_a)-F(a,x_a))| \underset{(<)}{\leqslant} 0, \tag{3.6}$$

where the left-hand side is equal to

$$\rho + (\lambda - 1)|f(x_a) - F(\hat{a}, x_a)| - \sigma(\hat{a}, x_a)(f_\lambda(x_a) - F(a, x_a)). \tag{3.7}$$

Since $\rho + (\lambda - 1)|f(x_a) - F(\hat{a}, x_a)|$ is positive for all $\lambda > 0$, $\sigma(\hat{a}, x_a)$ has to be equal to $\operatorname{sgn}(f_\lambda(x_a) - F(a, x_a))$ for all $\lambda > 0$ and different from zero. Therefore, (3.6) can be written as

$$\rho - \sigma(\hat{a}, x_a)(f(x_a) - F(\hat{a}, x_a)) + |f_\lambda(x_a) - F(\hat{a}, x_a)|$$
$$- |f_\lambda(x_a) - F(a, x_a)| \underset{(<)}{\leqslant} 0.$$

(iii) ⇒ (ii): Let $x_a$ be defined as in (iii). Then $\sigma(\hat{a}, x_a) = \operatorname{sgn}(f_\lambda(x_a) - F(\hat{a}, x_a))$ is different from zero and

$$\rho + (\lambda - 1)|f(x_a) - F(\hat{a}, x_a)| - |f_\lambda(x_a) - F(\hat{a}, x_a)$$
$$- (F(a, x_a) - F(\hat{a}, x_a))| \underset{(<)}{\leqslant} 0 \tag{3.8}$$

for all $\lambda > 0$. Next we show that (3.8) implies

$$\sigma(\hat{a}, x_a)(F(a, x_a) - F(\hat{a}, x_a)) > 0. \tag{3.9}$$

For assuming $\sigma(\hat{a}, x_a)(F(a, x_a) - F(\hat{a}, x_a)) > 0$, we get

$$\begin{aligned} &|f_\lambda(x_a) - F(a, x_a)|^2 \\ &\quad = |f_\lambda(x_a) - F(\hat{a}, x_a) - (F(a, x_a) - F(\hat{a}, x_a))|^2 \\ &\quad = (f_\lambda(x_a) - F(\hat{a}, x_a))^2 - 2\lambda(f(x_a) - F(\hat{a}, x_a))(F(a, x_a) - F(\hat{a}, x_a)) \\ &\qquad + (F(a, x_a) - F(\hat{a}, x_a))^2 < |f_\lambda(x_a) - F(\hat{a}, x_a)|^2, \end{aligned} \tag{3.10}$$

if

$$\lambda > \frac{|F(a, x_a) - F(\hat{a}, x_a)|}{2|f(x_a) - F(\hat{a}, x_a)|} = K;$$

if we insert further (3.10) into (3.8), we arrive at the contradiction

$$\rho + (\lambda - 1)|f(x_a) - F(\hat{a}, x_a)| < \lambda\, |f(x_a) - F(\hat{a}, x_a)|$$

for all $\lambda > K$. Hence (3.9) is true, and, since $\sigma(\hat{a}, x_a)$ is different from zero, we can rewrite (3.8) with (3.9) as

$$\rho + (\lambda - 1)|f(x_a) - F(\hat{a}, x_a)| - |\sigma(\hat{a}, x_a)(f_\lambda(x_a) - F(\hat{a}, x_a)) - \sigma(\hat{a}, x_a)(F(a, x_a) - F(\hat{a}, x_a))| \underset{(<)}{\leqslant} 0 \qquad \forall\, \lambda > 0.$$

Henceforth, condition (ii) holds true.

Considering the last proof we observe that the equivalence of Theorem 3.1 (ii) and (iii) is valid on arbitrary parameter sets $A \subseteq R^p$. If we further choose, in particular, $D = I(\hat{a})$, we can conclude the equivalence of the two conditions

(a) $\min_{x \in I(\hat{a})} \sigma(\hat{a}, x_a)(F(a, x_a) - F(\hat{a}, x_a)) \underset{(<)}{\leqslant} 0 \qquad \forall\, a \in A$

and

(b) For each $a \in A$ there is an $x_a \in I(\hat{a})$, independent of $\lambda$, with

$$|f_\lambda(x_a) - F(\hat{a}, x_a)| \underset{(<)}{\leqslant} |f_\lambda(x_a) - F(a, x_a)|$$

for all $\lambda > 0$ and $f_\lambda = F(\hat{a}) + \lambda(f - F(\hat{a}))$.

As is known, the general Kolmogoroff criterion (a) is also equivalent to

(c) $F(\hat{a})$ is a (strict) best approximation to $f_\lambda = F(\hat{a}) + \lambda(f - F(\hat{a}))$ for all $\lambda > 0$ with respect to $A$, i.e., $F(\hat{a})$ is a "solar point."

See, for example, [2]. (That "$<$" in (a) implies strictness in (c) and conversely, can be shown easily in the case of Chebyshev approximation.) Consequently, solar points are further characterized in a strong way by condition (b).

Based on these observations, we can consider condition (ii) of Theorem 3.1 as a generalization of the general Kolmogoroff criterion and condition (iii) as a generalization of a "local solar point" if we state

DEFINITION 3.1. $F(\hat{a})$ is called a "(strict) local solar point" if $F(\hat{a})$ is a (strict) local best approximation to $f_\lambda = F(\hat{a}) + \lambda(f - F(\hat{a}))$ for all $\lambda > 0$ with respect to a neighborhood $A \cap U_{\hat{a}}^\epsilon$ of $\hat{a}$ which is independent of $\lambda$.

With the last theorem we obtain

THEOREM 3.2. *Let*

(i) $\|f - F(\hat{a})\|_{I(\hat{a})} \underset{(<)}{\leqslant} \|f - F(a)\|_{I(\hat{a})} \qquad \forall\, a \in A \cap U_{\hat{a}}^\epsilon.$

(ii) $$\min_{x \in I(\hat{a})} \left\{ \sigma(\hat{a}, x)(F(a, x) - F(\hat{a}, x)) - \frac{(F(a, x) - F(\hat{a}, x))^2}{2\,\|f - F(\hat{a})\|} \right\} \underset{(<)}{\leqslant} 0 \qquad \forall\, a \in A \cap U_{\hat{a}}^\epsilon.$$

(iii) $\min_{x \in I(\hat{a})} \sigma(\hat{a}, x)(F(a, x) - F(\hat{a}, x)) \underset{(<)}{\leqslant} 0 \qquad \forall\, a \in A \cap U_{\hat{a}}^{\delta}$.

(iv) $F(\hat{a})$ is a strict local solar point in a neighborhood $A \cap U_{\hat{a}}^{\delta}$ of $\hat{a}$.

Then the following relations are true:

(i) $\Leftrightarrow$ (ii); (iii) $\Leftrightarrow$ (iv); (i), (ii) $\Rightarrow$ (iii), (iv) with a certain $\delta$, $0 < \delta \leqslant \varepsilon$; (iii), (iv) $\Rightarrow$ (i), (ii) with $\varepsilon = \delta$. Consequently, there is a neighborhood of $\hat{a}$ in which the four conditions above are equivalent. If further $B$ consists of finitely many points and if

$$\text{(v)} \qquad \|f - F(\hat{a})\| \underset{(<)}{\leqslant} \|f - F(a)\| \qquad \forall\, a \in A \cap U_{\hat{a}}^{\kappa},$$

then (i) $\Rightarrow$ (v) with $\kappa = \varepsilon$ and (v) $\Rightarrow$ (i) with a certain $\varepsilon$, $0 < \varepsilon \leqslant \kappa$. That means in the discrete case (i) to (v) are equivalent in a certain neighborhood of $\hat{a}$.

*Proof.* (i) $\Leftrightarrow$ (ii). Part (ii) is true if and only if the next inequality is valid:

$$\min_{x \in I(\hat{a})} \{2(f(x) - F(\hat{a}, x))(F(a, x) - F(\hat{a}, x)) - (F(a, x) - F(\hat{a}, x))^2\}$$

$$= \min_{x \in I(\hat{a})} \{2(f(x) - F(\hat{a}, x))^2 - 2(f(x) - F(\hat{a}, x))(f(x) - F(a, x))$$

$$- (f(x) - F(\hat{a}, x))^2 + 2(f(x) - F(\hat{a}, x))(f(x) - F(a, x))$$

$$- (f(x) - F(a, x))^2\}$$

$$= \min_{x \in I(\hat{a})} \{(f(x) - F(\hat{a}, x))^2 - (f(x) - F(a, x))^2\} \underset{(<)}{\leqslant} 0 \qquad \forall\, a \in A \cap U_{\hat{a}}^{\varepsilon}.$$

(iii) $\Leftrightarrow$ (iv) was discussed above.

(i) $\Rightarrow$ (iii) with a certain $\delta$, $0 < \delta \leqslant \varepsilon$, and (iii) $\Rightarrow$ (i) with $\varepsilon = \delta$ follows from Theorem 3.1 if we choose $D = I(\hat{a})$.

(i) $\Rightarrow$ (v) was proved with Lemma 2.2.

Finally, we show under the assumption $B$ consists of finitely many points that (v) implies (i) for a certain $\varepsilon$, $0 < \varepsilon \leqslant \kappa$:

Since for $I(\hat{a}) = B$ the implication is obvious, we can assume that there is an $x_i \in B \backslash I(\hat{a})$ so that

$$\max_{x_i \in B \backslash I(\hat{a})} |f(x_i) - F(\hat{a}, x_i)| = C < \|f - F(\hat{a})\| = \rho$$

is achieved. Because of the continuity of the mapping $F: A \to C(B)$ there exists now a $\delta_{\hat{\varepsilon}}$ for $\hat{\varepsilon} = \rho - C$ such that

$$\|F(a) - F(\hat{a})\| < \rho - C \qquad \forall\, a \in A \cap U_{\hat{a}}^{\delta_{\hat{\varepsilon}}}.$$

This proves the assumption for $\delta = \min(\varepsilon, \delta_{\bar{\varepsilon}})$; for if we assume that there is an $x_j \in I(a)$, $a \in A \cap U_{\hat{a}}^{\delta}$, and $x_j \notin I(\hat{a})$, we obtain the contradiction

$$\|f - F(\hat{a})\| \leqslant \|f - F(a)\| = |f(x_j) - F(\hat{a}, x_j) - (F(a, x_j) - F(\hat{a}, x_j))| < C + \rho - C = \rho.$$

By the first part of the last theorem, results of [4] and [6], which we referred to in the introduction, are joined and proved again where we showed in addition that strictness in one condition implies always strictness in the other condition, respectively. Beyond this, Theorem 3.2 states that in the discrete case: each local best approximation to $f$ on $B$ is also a local best approximation to $f$ on the set of extremal points, and is, furthermore, characterized by the general Kolmogoroff criterion. It might be possible to use these results to advantage numerically.

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